q-alg/9710009

# QUANTUM ORTHOGONAL CALEY-KLEIN GROUPS AND ALGEBRAS

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The extension of FRT quantization theory for the nonsemisimple CK groups is suggested. The quantum orthogonal CK groups are realized as the Hopf algebras of the noncommutative functions over an associative algebras with nilpotent commutative generators. The quantum CK algebras are obtained as the dual objects to the corresponding quantum groups.

## 1 Introduction

The quantization theory of the simple groups and algebras Lie was developed by Faddeev-Reshetikhin-Takhtadjan (FRT) [1]. In group theory there is a remarkable set of groups, namely the motion groups of n-dimensional spaces of constant curvature or the orthogonal Cayley-Klein (CK) groups. The well known Euclidean $E(n)$, Poincare $P(n)$, Galileian $G(n)$ and other nonsemisimple groups are in the set of CK groups. The principal proposal for quantization of all CK groups is to regard them as the groups over an algebra $\mathbf{D}$ with nilpotent commutative generators and the corresponding quantum CK groups as the algebra of noncommutative functions over $\mathbf{D}$.

Algebra $\mathbf{D}_n(\iota;\mathbf{C})$ is defined as an associative algebra with unit and *nilpotent* generators $\iota_1,\ldots,\iota_n,\ \iota_k^2=0,\ k=1,\ldots,n$ with *commutative* multiplication $\iota_k\iota_m=\iota_m\iota_k,\ k\neq m$. The general element of $\mathbf{D}_n(\iota;\mathbf{C})$ has the form $a=a_0+\sum_{p=1}^{n}\sum_{k_1<\ldots<k_p} a_{k_1\ldots k_p}\iota_{k_1}\ldots\iota_{k_p},\quad a_0,a_{k_1\ldots k_p}\in\mathbf{C}$. For $n=1$ we have $\mathbf{D}_1(\iota_1;\mathbf{C})\ni a=a_0+a_1\iota_1$, i.e. dual (or Study) numbers. For $n=2$ the general element of $\mathbf{D}_2(\iota_1,\iota_2;\mathbf{C})$ is written as follows: $a=a_0+a_1\iota_1+a_2\iota_2+a_{12}\iota_1\iota_2$.

## 2 Orthogonal CK groups $SO(N;j;\mathbf{R})$

Let us regard a vector space $\mathbf{R}_N(j)$ over $\mathbf{D}_{N-1}(j;\mathbf{R})$ with Cartesian coordinates $x(j)=(x_1,J_{12}x_2,\ldots,J_{1,N}x_N)^t,\ x_k\in\mathbf{R},\ k=1,\ldots,N$ and quadratic form $x^t(j)x(j)=x_1^2+\sum_{k=2}^{N}J_{1k}^2x_k^2$, where $J_{\mu\nu}=\prod_{r=\mu}^{\nu-1}j_r,\ \mu<\nu,\ J_{\mu\nu}=1,\ \mu\geq\nu,\ j_r=1,\iota_r,i$.

Orthogonal CK groups $SO(N;j;\mathbf{R})$ are defined as the set of transformations of $\mathbf{R}_N(j)$ leaving invariant $x^t(j)x(j)$ and are realized in the Cartesian

basis as the matrix groups over $\mathbf{D}_{N-1}(j;\mathbf{R})$ with the help of the *special* matrices

$$(A(j))_{kp}=\tilde{J}_{kp}a_{kp},\quad a_{kp}\in\mathbf{R},\quad \tilde{J}_{kp}=J_{kp},\ k<p,\quad \tilde{J}_{kp}=J_{pk},\ k\geq p, \tag{1}$$

These matrices act on vectors $x(j)\in\mathbf{R}_N(j)$ by matrix multiplication and are satisfied the following $j$-orthogonality relations: $A(j)A^t(j)=A^t(j)A(j)=I$.

One of the solutions of matrix equation $DC_0D^t=I,\ (C_0)_{ik}=\delta_{ik'},\ k'=N+1-k$ provide the similarity transformation $B(j)=D^{-1}A(j)D$ which give the realization of $SO_q(N;\mathbf{C})$ in a new ("symplectic") basis with invariant quadratic form $x^t(j)C_0x(j)$ and the additional relations of $j$-orthogonality $B(j)C_0B^t(j)=B^t(j)C_0B(j)=C_0$.

## 3 Quantum groups and algebras

We shall regard the quantum deformations of the contracted CK groups, i.e. $j_k=1,\iota_k$. We shall start with the $\mathbf{D}\langle t_{ik}\rangle$ — the algebra of noncommutative polynomials of $N^2$ variables $t_{ik},\ i,k=1,\ldots,N$ over the algebra $\mathbf{D}_{N-1}(j)$. In addition we shall transform the deformation parameter $q=\exp z$ as follows: $z=Jv,\ J\equiv J_{1N}=\prod_{k=1}^{N-1}j_k$, where $v$ is the new deformation parameter.

In "symplectic" basis the quantum CK group $SO_v(N;j;\mathbf{C})$ is produced by the generating matrix $T(j)\in M_N(\mathbf{D}\langle t_{ik}\rangle)$ equal to $B(j)$ for $q=1$. The noncommutative entries of $T(j)$ obey the commutation relations

$$R_v(j)T_1(j)T_2(j)=T_2(j)T_1(j)R_v(j). \tag{2}$$

and the additional relations of $(v,j)$-orthogonality

$$T(j)C(j)T^t(j)=T^t(j)C(j)T(j)=C(j), \tag{3}$$

where lower triangular R–matrix $R_v(j)$ and $C(j)$ are obtained from $R_q$ and $C$, respectively, by substitution $Jv$ instead of $z: R_v(j)=R_q(z\rightarrow Jv),\quad C(j)=C(z\rightarrow Jv)$. Then the quotient

$$SO_v(N;j;\mathbf{C})=\mathbf{D}\langle t_{ik}\rangle\big/(2),(3) \tag{4}$$

is Hopf algebra with the coproduct $\Delta$, counit $\epsilon$ and antipode $S:$

$$\Delta T(j)=T(j)\dot{\otimes}T(j),\quad \epsilon(T(j))=I,\quad S(T(j))=C(j)T^t(j)C^{-1}(j). \tag{5}$$

By FRT quantization theory [1] the dual space $Hom(SO_v(N;j;\mathbf{C}),\mathbf{C})$ is an algebra with the multiplication induced by coproduct $\Delta$ in $SO_v(N;j;\mathbf{C})$

$$l_1l_2(a)=(l_1\otimes l_2)(\Delta(a)), \tag{6}$$

$l_1, l_2 \in Hom(SO_v(N;j;\mathbf{C}),\mathbf{C}), \quad a \in SO_v(N;j)$. Let us formally introduce $N \times N$ upper $(+)$ and lower $(-)$ triangular matrices $L^{(\pm)}(j)$ as follows: it is necessary to put $j_k^{-1}$ in the nondiagonal matrix elements of $L^{(\pm)}(j)$, if there is the parameter $j_k$ in the corresponding matrix element of $T(j)$. For example, if $(T(j))_{12} = j_1 t_{12} + j_2 \tilde{t}_{12}$, then $(L^{(+)}(j))_{12} = j_1^{-1} l_{12} + j_2^{-1} \tilde{l}_{12}$. Formally the matrices $L^{(\pm)}(j)$ are not defined for $j_k = \iota_k$, since $\iota_k^{-1}$ do not exist, but if we set an action of the matrix functionals $L^{(\pm)}(j)$ on the elements of $SO_v(N;j;\mathbf{C})$ by the duality relation

$$\langle L^{(\pm)}(j), T(j) \rangle = R^{(\pm)}(j), \tag{7}$$

where $R^{(+)}(j) = PR_v(j)P, \ R^{(-)}(j) = R_v^{-1}(j), \ Pu \otimes w = w \otimes u$, then we shall have well defined expressions even for $j_k = \iota_k$.

The elements of $L^{(\pm)}(j)$ satisfy the commutation relations

$$\begin{aligned} R^{(+)}(j) L_1^{(\sigma)}(j) L_2^{(\sigma)}(j) &= L_2^{(\sigma)}(j) L_1^{(\sigma)}(j) R^{(+)}(j), \\ R^{(+)}(j) L_1^{(+)}(j) L_2^{(-)}(j) &= L_2^{(-)}(j) L_1^{(+)}(j) R^{(+)}(j), \quad \sigma = \pm \end{aligned} \tag{8}$$

and additional relations

$$\begin{aligned} L^{(\pm)}(j) C^t(j) L^{(\pm)}(j) &= C^t(j), \\ L^{(\pm)}(j) (C^t(j))^{-1} L^{(\pm)}(j) &= (C^t(j))^{-1}, \\ l_{kk}^{(+)} l_{kk}^{(-)} = l_{kk}^{(-)} l_{kk}^{(+)} = 1, \qquad & l_{11}^{(+)} \ldots l_{NN}^{(+)} = 1, \ k = 1, \ldots, N. \end{aligned} \tag{9}$$

An algebra $so_v(N;j;\mathbf{C}) = \{I, L^{(\pm)}(j)\}$ is called quantum CK algebra and is Hopf algebra with the following coproduct $\Delta$, antipode $S$ and counit $\epsilon$ : $\Delta L^{(\pm)}(j) = L^{(\pm)}(j) \dot{\otimes} L^{(\pm)}(j), \quad S(L^{(\pm)}(j)) = C^t(j) (L^{(\pm)}(j))^t (C^t(j))^{-1}$, $\epsilon(L^{(\pm)}(j)) = I$.

It is possible to show that algebra $so_v(N;j;\mathbf{C})$ is isomorphic with the quantum deformation [3] of the universal enveloping algebra of the CK algebra $so(N;j;\mathbf{C})$, which may be obtained from the orthogonal algebra $so(N;\mathbf{C})$ by contractions [2].

## References


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